The generalized Coulomb problem: an exactly solvable model

1993 J. Phys. A: Math. Gen. 26 3301

(http://iopscience.iop.org/0305-4470/26/13/032)

J. Phys. A: Math. Gen. 26 (1993) 3301–3306. Printed in the UK

The generalized Coulomb problem: an exactly solvable model

Géza Lévai†§ and Brian Wesley Williams‡

† Sloane Physics Laboratory, Yale University, PO Box 6666, New Haven, CT 06511, USA

‡ Department of Chemistry, Bucknell University, Lewisburg, PA 17837, USA

Received 12 February 1993

Abstract. We investigate an exactly solvable potential class which contains the Coulomb potential as a special case. These potentials have an angular-momentum-dependent repulsive core, but retain several important characteristics of the Coulomb problem. Some possible fields of their application are also proposed.

The past ten years have seen revived interest in exactly solvable potentials of non-relativistic quantum mechanics, partly as the result of the introduction of supersymmetric quantum mechanics (SUSYQM) [1]. These investigations have focused mainly on the fundamental aspects (classification, symmetries, etc) of solvable potentials and on the identification of new potentials of this kind. Although the general solution of large potential families have been given, the resulting formulae are usually too complicated for practical use. Some of the special subclasses, however, may avoid these drawbacks, and at the same time may offer potential shapes different from those of the simplest solvable problems. Here we describe one more of these 'non-trivial' potentials and suggest possible fields of its application.

The transformation reducing the one-dimensional Schrödinger equation

$$\frac{\mathrm{d}^2\Psi}{\mathrm{d}x^2} + (E - V(x))\Psi(x) = 0 \tag{1}$$

to the second-order differential equation

$$\frac{\mathrm{d}^2 F}{\mathrm{d}g^2} + Q(g)\frac{\mathrm{d}F}{\mathrm{d}g} + R(g)F(g) = 0 \tag{2}$$

of some special function $F(g)$ can be formulated in various ways. The most general procedure of this kind for the hypergeometric and confluent hypergeometric functions was given by Natanzon [2]. The general Natanzon and Natanzon confluent [3] potentials are written as complicated six-parameter expressions of the coordinate, and their energy eigenvalues E_n are given by an implicit formula.

This method allows the general treatment of a large class of potentials with a wide variety of shapes; nevertheless their extensive application is usually hindered by technical difficulties in their most general form. Until now only some special subclasses have been studied in detail. In addition to the simplest such subclass, the 'shape-invariant' potentials [4] (which contain many of the most well known textbook examples of exactly solvable

§ Permanent address: Institute of Nuclear Research of the Hungarian Academy of Sciences, Debrecen, PO Box 51, Hungary 4001.

0305-4470/93/133301+06$07.50

potentials), only a few other cases (for example the Ginocchio [5], the 'PIII' [6], and some other potentials [7, 8]) have been investigated more or less thoroughly. A common feature of most of these potentials is that they occupy an in-between situation between the trivial shape-invariant potentials and the general Natanzon (confluent) potentials. In particular, the potentials in [5] and [6] correspond to Natanzon potentials with zero value of one of the three parameters determining the variable transformation $g(x)$ in equations (1) and (2), which choice also simplifies the determination of the energy eigenvalues from the implicit formula but, at the same time, results in an implicit function $x(g)$ rather than $g(x)$. The potentials in [8] represent another kind of special case, in some sense opposite to the previous examples, as the three parameters have the same non-zero value, which results in a complicated energy formula, but helps to avoid having an implicit $x(g)$ function. (The 12 known shape-invariant potentials correspond to cases in which two of these parameters are zero and therefore both their energy formula and $g(x)$ have relatively simple forms.)

Here we describe one more of these 'intermediate' potentials, which are more general than the shape-invariant ones, but at the same time can be handled in a relatively simple way. In contrast with most of the previously mentioned potentials [5, 6, 8] which were obtained from the hypergeometric equation, this one belongs to the Natanzon confluent class [3], in fact it is the generalization of one of the potentials discussed briefly in [7]. It can also be considered the generalized version of the Coulomb problem, which it contains as a special (shape-invariant) subcase. (There is a similar relationship between the Ginocchio [5] and the Pöschl–Teller potentials.) In order to derive it, here we use an old method [9] which can simply be related to algebraic techniques [10] and can be combined [11] with the formalism of supersymmetric quantum mechanics.

Substituting $\Psi(x) = f(x)F(g(x))$ in (1) we find that (2) is recovered if

$$E - V(x) = (g'(x))^2 R(g(x)) - \left[\left(\frac{f'(x)}{f(x)}\right)^2 + \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f'(x)}{f(x)}\right)\right]$$
$$= \frac{g'''(x)}{2g'(x)} - \frac{3}{4}\left(\frac{g''(x)}{g'(x)}\right)^2 + (g'(x))^2\left(R(g(x)) - \frac{1}{2}\frac{\mathrm{d}Q}{\mathrm{d}g} - \frac{1}{4}Q^2(g(x))\right) \tag{3}$$

holds, where

$$f(x) \simeq (g'(x))^{-1/2}\exp\left(\frac{1}{2}\int^{g(x)} Q(g)\,\mathrm{d}g\right). \tag{4}$$

Now, considering specifically the confluent hypergeometric function $F(-n, \beta; g(x))$ and introducing the simple $g(x) = \rho h(x)$ substitution we get

$$E_n - V(x) = \frac{h'''(x)}{2h'(x)} - \frac{3}{4}\left(\frac{h''(x)}{h'(x)}\right)^2$$
$$+ \frac{(h'(x))^2}{h(x)}\rho\left(n + \frac{\beta}{2}\right) - (h'(x))^2\frac{\rho^2}{4} + \frac{(h'(x))^2}{(h(x))^2}\frac{\beta}{2}\left(1 - \frac{\beta}{2}\right). \tag{5}$$

Observing that a constant (x-independent) term has to be present on the right-hand side to account for E_n on the left-hand side we can set up differential equations to determine $h(x)$. Selecting one of the last three terms on the right-hand side of (5) the three shape-invariant potentials of the confluent hypergeometric case (the three-dimensional harmonic oscillator, the Coulomb problem and the Morse potential) could be recovered. Considering the combination of the first two such terms by

$$(h'(x))^2\left(1+\frac{\theta}{h(x)}\right)=C \tag{6}$$

and solving the differential equation

$$\frac{\mathrm{d}h}{\mathrm{d}x}=C^{1/2}\left(\frac{h}{h+\theta}\right)^{1/2} \tag{7}$$

we obtain $x(h)$ rather than $h(x)$ as

$$C^{1/2}x=\theta\tanh^{-1}\left[\left(\frac{h}{h+\theta}\right)^{1/2}\right]+(h(h+\theta))^{1/2} \tag{8}$$

which is similar to the situation with most of the other exactly solvable non-shape-invariant potentials [5–7].

This implicit $h(x)$ function can, of course, be computed to any desired accuracy and therefore the potential $V(x)=V(h(x))$ and the wavefunctions $\Psi(x)=\Psi(h(x))$ can also be evaluated. $h(x)$ maps the $[0,\infty)$ half-axis onto itself, its asymptotic behaviour for $x\to\infty$ is $h(x)\to C^{1/2}x$, while the series expansion of the right-hand side of (8) reveals that it can be approximated by $Cx^2/4\theta$ near the origin. From here on we shall consider $V(x)$ a central potential and replace x with r.

Substituting this $h(r)$ into (5) we get

$$\begin{aligned}E_n-V(x)=&-\frac{C}{4}\rho^2+\left[\frac{\rho^2\theta}{4}+\rho\left(n+\frac{\beta}{2}\right)\right]\frac{C}{h(r)+\theta}-\left(\frac{\beta}{2}-1\right)\frac{\beta}{2}\frac{C}{h(r)(h(r)+\theta)}\\&-\frac{C\theta}{2(h(r)+\theta)^3}-\frac{3C\theta^2}{16h(r)(h(r)+\theta)^3}.\end{aligned} \tag{9}$$

The n-dependence of the potential term can be removed by introducing the constant

$$D=\frac{\theta}{4}\rho^2+\rho\left(n+\frac{\beta}{2}\right)=\text{constant} \tag{10}$$

which amounts to a specific choice of $\rho\equiv\rho_n$. Rewriting the formulae in this way, we obtain the final expression of the wavefunctions

$$\Psi_n(r)=N(h(r)+\theta)^{1/4}(h(r))^{(2\beta-1)/4}\exp\left(-\frac{\rho_n}{2}h(r)\right)F(-n,\beta;\rho_nh(r)) \tag{11}$$

the potential

$$\begin{aligned}V(r)=&-\frac{CD}{h(r)+\theta}+\left(\frac{\beta}{2}-1\right)\frac{\beta}{2}\frac{C}{h(r)(h(r)+\theta)}+\frac{C\theta}{2(h(r)+\theta)^3}\\&+\frac{3C\theta^2}{16h(r)(h(r)+\theta)^3}\end{aligned} \tag{12}$$

and the energy eigenvalues

$$E_n=-\frac{C}{4}\rho_n^2=-\frac{C}{\theta^2}\left\{\left[\left(n+\frac{\beta}{2}\right)^2+D\theta\right]^{1/2}-\left(n+\frac{\beta}{2}\right)\right\}^2. \tag{13}$$

(Here we chose the zero point of the energy scale at $V(r \to \infty) = 0$.) The form of the wavefunction in (11) suggests that it can be regular only for positive values of ρ_n, therefore only the positive root of (10) should be considered. The normalization factor N in (11) can be determined by performing straightforward integrations [12]:

$$N = \frac{C^{1/4}\rho_n^{(\beta+1)/2}}{\Gamma(\beta)}\left(\frac{\Gamma(n+\beta)}{n!(\beta+2n+\theta\rho_n)}\right)^{1/2}. \tag{14}$$

β is restricted to positive values only, and here we consider only $\theta \geqslant 0$ to avoid singularities for $r > 0$. In this case D can also have only positive values. Similarly to the Coulomb problem, any non-negative integer values of n are allowed, therefore this potential supports an infinite number of bound states, with energy eigenvalues E_n converging to zero from below in the $n \to \infty$ limit. Taking $\theta = 0$, all the above expressions, of course, yield the corresponding results for the Coulomb problem with $2\mu Ze^2/\hbar^2 = C^{1/2}D$ and $l = -1+\beta/2$. (However, some care should be taken in recovering the standard results of the hydrogen atom, as the generalized Laguerre polynomials normally used in these formulae are defined in a different way in mathematical handbooks [12] and in most of the textbooks on quantum mechanics.)

The approximate behaviour of the four terms of the potential $V(r)$ near the origin and in the $r \to \infty$ limit is summarized in table 1. These results suggest that this potential can be considered a Coulomb potential in three spatial dimensions distorted by an angular-momentum-dependent repulsive central potential (if we interpret β as $2(l+1)$). In this case it can be rewritten as

$$V(r) = V_0(r) + V_l(r) + \frac{l(l+1)}{r^2} \tag{15}$$

Table 1. Approximate behaviour of the four terms of the potential in (12) near the origin and in the asymptotic limit.

		$-\frac{CD}{h(r)+\theta}$	$\left(\frac{\beta}{2}-1\right)\frac{\beta}{2}\frac{C}{h(r)(h(r)+\theta)}$	$\frac{C\theta}{2(h(r)+\theta)^3}$	$\frac{3C\theta^2}{16h(r)(h(r)+\theta)^3}$
$r \to 0$	$1\times$	$-\frac{CD}{\theta}$	—	$\frac{C}{2\theta^2}$	—
	$\frac{1}{r^2}\times$	—	$4\left(\frac{\beta}{2}-1\right)\frac{\beta}{2}$	—	$\frac{3}{4}$
$r \to \infty$	$\frac{1}{r}\times$	$-C^{1/2}D$	—	—	—
	$\frac{1}{r^2}\times$	$D\theta$	$\left(\frac{\beta}{2}-1\right)\frac{\beta}{2}$	—	—
	$\frac{1}{r^3}\times$	$-\frac{D\theta^2}{C^{1/2}}$	$-\frac{\theta}{C^{1/2}}\left(\frac{\beta}{2}-1\right)\frac{\beta}{2}$	$\frac{\theta}{2C^{1/2}}$	—
	$\frac{1}{r^4}\times$	$\frac{D\theta^3}{C}$	$\frac{\theta^2}{C}\left(\frac{\beta}{2}-1\right)\frac{\beta}{2}$	$-\frac{3\theta^2}{2C}$	$\frac{3\theta^2}{16C}$

where

$$V_0(r) = -\frac{CD}{h(r)+\theta} + \frac{C\theta}{2(h(r)+\theta)^3} + \frac{3C\theta^2}{16h(r)(h(r)+\theta)^3} \tag{15a}$$

and

$$V_l(r) = \frac{l(l+1)}{r^2}\left(\frac{Cr^2}{h(r)(h(r)+\theta)} - 1\right) \equiv \frac{l(l+1)}{r^2}v(r)\,. \tag{15b}$$

One important implication of these results is that the above potential extends the limited range of central potentials which are exactly solvable for any value of the angular momentum l. $V_0(r)$ is Coulomb-like for large r and has a singular repulsive 'core' $V_0(r) \simeq 3/4r^2$ near the origin. The angular-momentum-dependent term $V_l(r)$ is also repulsive, and decreases monotonously with increasing values of r. This is partly due to $v(r)$ which also decreases monotonously (from the value of 3 to 0), decaying more rapidly for small values of θ (and, of course, disappearing in the $\theta = 0$ (Coulomb) case). In figure 1, potential (15) has been plotted for $l = 0$, 1 and 2 (i.e. $\beta = 2$, 4 and 6) using two different values of θ.

Quantum mechanical potentials with properties described above could be used to simulate effective forces arising from electrons occupying inner atomic shells. As an early application of supersymmetric quantum mechanics, for example, similar ($1/r^2$-like) repulsive potentials have been used in [13] to inter-relate different alkali metal atoms within the framework of phenomenologic supersymmetry. Considering another length scale, potential (15) could also be used in the description of mesonic atoms to account for the deviations from the unperturbed Coulomb potential due to the finite size of nuclei, which have been approximated previously by the 'cut-off' Coulomb potential [14], for example.

Similarly to the Coulomb problem, the energy eigenvalues of the bound states depend on $n + \beta/2$, i.e. $n + l + 1$, therefore this potential exhibits a level degeneracy similar to that of the Coulomb problem, which is related to an $O(4)$ symmetry group. An interesting task would be to inspect the symmetry underlying this degeneracy, but it is beyond the scope of this work.

In terms of SUSYQM, $V(r)$ and E_n can be obtained from the superpotential

$$W(r) = \frac{C^{1/2}}{(h(r)(h(r)+\theta))^{1/2}}\left(\frac{\theta}{4(h(r)+\theta)} - \frac{\beta}{2} + \frac{\rho_0}{2}h(r)\right). \tag{16}$$

Deriving the supersymmetric partner of $V_-(r) \equiv V(r) - E_0$,

$$V_+(r) = W^2(r) + \frac{\mathrm{d}W}{\mathrm{d}x} \tag{17}$$

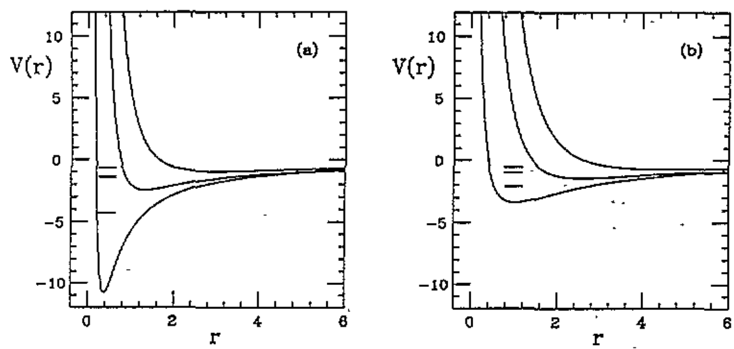

Figure 1. Potential (15) plotted for $\theta = 0.2$ (*a*) and 1.0 (*b*), keeping C and D at constant values ($C = 1.0$ and $D = 5.0$). We have displayed the potential curves for $l = 0$, 1 and 2 (i.e. $\beta = 2$, 4 and 6) together with the position of the lowest-lying energy level for each partial wave l.

we find that similarly to the other 'non-trivial' exactly solvable potentials [5–8] this one is also outside the shape-invariant class. Furthermore, the wavefunctions of $V_+(r)$ can be written as the sum of two confluent hypergeometric functions, which indicates [15] that it is outside the Natanzon confluent class and belongs to an even more general family of solvable potentials. It is worth mentioning, though, that $V_+(r)$ with $\beta = 2(l+1)$ has the *same* number of states with the *same* energy eigenvalues as $V_-(r)$ has with $\beta+2 = 2(l+2)$, but still the two potentials and the corresponding wavefunctions are different, and become identical only in the $\theta = 0$ shape-invariant (Coulomb) limit.

Finally, we show how this potential can be derived from other approaches. It can be obtained from the general Natanzon confluent potential in [3] after selecting the following set of parameters: $\sigma_2 = 4/C$, $\sigma_1 = 4\theta/C$, $c_0 = 0$; $g_2 = 0$, $g_1 = -4D$, $\eta = (\beta+1)^2$. It also corresponds to the generalized version of the type-(d) 'implicit potential' in table 1 of [7] derived from one realization of the $SU(1,1)$ spectrum-generating algebra.

In summary, we have described a class of potentials which can be interpreted as the generalization of the Coulomb problem. It has an angular-momentum-dependent repulsive 'core', but it is solvable for any l. This example emphasizes the importance of those special subclasses of the Natanzon (confluent) potentials, which can be obtained by a similar generalization of other shape-invariant potentials. These potentials retain certain features of the simple shape-invariant potentials, but offer a wider variety of shapes and therefore may find more widespread applications.

Acknowledgments

The authors are grateful to one of the referees for pointing out some inaccuracies in the formulae. GL thanks the IIE Fulbright Program for a research fellowship, and extends his sincere thanks to Professor F Iachello for his hospitality at Yale University. BWW would like to thank Bucknell University for an untenured leave permitting this research. This work was also partly supported by the OTKA Grant No F4303 (Hungary).

References

[1] Witten E 1981 *Nucl. Phys.* B **188** 513
[2] Natanzon G A 1979 *Teor. Mat. Fiz.* **38** 146
[3] Cordero P and Salamó S 1991 *J. Phys. A: Math. Gen.* **24** 5299
[4] Gendenshtein L E 1983 *Zh. Eksp. Teor. Fiz. Pis. Red.* **38** 299 (Engl. Transl. *JETP Lett.* **38** 356)
[5] Ginocchio J N 1984 *Ann. Phys., NY* **152** 203
[6] Lévai G 1991 *J. Phys. A: Math. Gen.* **24** 131
[7] Brajamani S and Singh C A 1990 *J. Phys. A: Math. Gen.* **23** 3421
[8] Singh C A and Devi T H 1992 *Phys. Lett.* **171A** 249
[9] Bhattacharjie A and Sudarshan E C G 1962 *Nuovo Cimento* **25** 864
[10] Cordero P, Hojman S, Furlan P and Ghirardi G C 1971 *Nuovo Cimento* A **3** 807
[11] Lévai G 1989 *J. Phys. A: Math. Gen.* **22** 689
[12] Abramowitz M and Stegun I 1970 *Handbook of Mathematical Functions* (New York: Dover)
[13] Kostelecký V A and Nieto M M 1985 *Phys. Rev.* A **32** 1293
[14] Sinha A and Roychoudhury R 1990 *J. Phys. A: Math. Gen.* **23** 3869
[15] Cooper F, Ginocchio J N and Khare A 1987 *Phys. Rev.* D **36** 2458